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Translations in simply transitive affine actions of 5-dimensional nilpotent Lie groups

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Abstract

In this paper we determine all 5-dimensional nilpotent Lie groups acting simply transitively and affinely on $\mathbb{R}^5$ in such a way that only the identity element acts as a pure translation. In fact, a complete classification of all possible such actions is obtained by translating the original problem in terms of complete left symmetric structures on the corresponding Lie algebra. It turns out that some 5-dimensional nilpotent Lie groups admit no such actions, while others admit a finite number of such actions, and still others even admit infinite families of such actions.

1. Introduction

In this paper we study simply transitive affine actions of Lie groups on some space $\mathbb{R}^n$. It is well known that any Lie group admitting such a simply transitive affine action must be solvable [1]. The converse question, due to Milnor [12], asking whether any simply connected and connected solvable Lie group admits a simply transitive affine action has been answered negatively by Y. Benoist, even in the nilpotent case ([2], see also [3,4]). Interesting questions now are: "Which (nilpotent) Lie groups do admit such an action?" and "If a Lie group admits such an action, is it possible to classify all these actions?".

An important conjecture (whose truth implies a possible classification of all simply transitive affine actions of a given nilpotent Lie group in an iterative way) was the one formulated by Auslander [1]. This conjecture stated that any nilpotent simply transitive group of affine motions must contain a one-parameter group of pure translations in its

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centre. However, Fried [7] was able to find a counter example, namely there exists a simply transitive affine action of the 3-step nilpotent 4-dimensional connected and simply connected nilpotent Lie group where the only element acting as a pure translation is the identity element. In fact, in [10], H. Kim showed that, up to affine equivalence, there are exactly 2 simply transitive affine actions without translations of this Lie group. On the other hand Fried [7] also proved that for Abelian Lie groups the conjecture is valid [7]. Recently, Medina and Khakimdjanov [11] showed that for odd dimensional filiform Lie groups the Auslander conjecture does hold, while for every even $n \geqslant 4$ they extended the example of D. Fried and constructed a simply transitive affine action without translation on the “standard” filiform Lie group of dimension n (filiform meaning n-dimensional and nilpotent of class $n-1$). In [6] we have been studying the case of nilpotency class 2. We obtained that for 2-step nilpotent Lie groups with a one-dimensional commutator subgroup (Heisenberg type Lie groups) the Auslander conjecture does hold. This result cannot be extended to groups with a higher dimensional commutator subgroup. (We found an example of a 5-dimensional nilpotent Lie group with a 2-dimensional commutator subgroup for which the Auslander conjecture does not hold.)

These results imply that the only 4-dimensional nilpotent Lie group acting simply transitively, affinely and without pure translations is the unique filiform one in dimension 4 (for which we know that there are 2 non-equivalent such actions). This result was already known, as H. Kim was able to classify all simply transitive and affine actions of 4-dimensional nilpotent Lie groups [10].

In this paper we study the five-dimensional case. In order to understand this situation, there are two types of simply transitive affine actions: those with non-trivial translations, and those without. The first situation is in a certain sense the easiest one to understand, since the study of these actions can be reduced to the lower dimensional cases, using cohomological techniques. For the second situation, the simply transitive actions without non-trivial translations, there is no general technique available, and these cases have to treated separately for each Lie group.

In this paper we solve the second situation for all 5-dimensional connected and simply connected nilpotent Lie groups. So, for each Lie group we are investigating whether they admit simply transitive affine actions without translations. And, if so, we also want to give a classification of all these actions.

Although the problem is formulated on the Lie group level, we shall treat it at the Lie algebra level. There is indeed a very nice correspondence between simply transitive affine actions of a nilpotent Lie group and the so-called complete left symmetric structures on the corresponding Lie algebra. Simply transitive affine actions of a nilpotent Lie group without translations will correspond to complete left symmetric structures with trivial centre (see Section 2). So the main topic of this paper is in fact the study and classification of all centerless complete left symmetric structures on 5-dimensional real nilpotent Lie algebras.

In order to be able to summarize the results obtained, let us first give a classification of all five-dimensional nilpotent Lie algebras over $\mathbb{R}$.

Theorem 1.1 (See [13, p. 210]). *Let $\mathfrak{g}$ be a five-dimensional nilpotent Lie algebra over $\mathbb{R}$, with vector space basis $\langle X_1, X_2, X_3, X_4, X_5\rangle$, then $\mathfrak{g}$ is isomorphic to one* (*and only one*) *of the following Lie algebras*: (*we only write the non-zero brackets*!)

- $\mathfrak{g}_1 = \mathbb{R}^5$;
- $\mathfrak{g}_2$: $[X_1, X_2] = X_3$;
- $\mathfrak{g}_3$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$;
- $\mathfrak{g}_4$: $[X_1, X_2] = X_5$, $[X_3, X_4] = X_5$;
- $\mathfrak{g}_5$: $[X_1, X_2] = X_4$, $[X_1, X_3] = X_5$;
- $\mathfrak{g}_6$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_5] = X_4$;
- $\mathfrak{g}_7$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$;
- $\mathfrak{g}_8$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$;
- $\mathfrak{g}_9$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_1, X_4] = X_5$, $[X_2, X_3] = X_5$.

$\mathfrak{g}_1$ is the trivial Lie algebra of dimension 5, $\mathfrak{g}_2 = \mathfrak{h}_1 \oplus \mathbb{R}^2$ with $\mathfrak{h}_1$ the Heisenberg Lie algebra of dimension 3, $\mathfrak{g}_4 = \mathfrak{h}_2$, the Heisenberg Lie algebra of dimension 5, $\mathfrak{g}_7$ is the free 3-step nilpotent Lie algebra on 2 generators, $\mathfrak{g}_8$ and $\mathfrak{g}_9$ are two 5-dimensional Lie algebras which are filiform (nilpotent of class 4).

In this paper we will show the following:

Main result:

- $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_4, \mathfrak{g}_7, \mathfrak{g}_8$ and $\mathfrak{g}_9$ do not admit a centerless complete left symmetric structure.
- $\mathfrak{g}_5$ admits, up to isomorphism, 5 complete left symmetric structures with trivial centre.
- $\mathfrak{g}_3$ admits, up to isomorphism, 6 complete left symmetric structures with trivial centre.
- $\mathfrak{g}_6$ admits, up to isomorphism, infinitely many complete left symmetric structures with trivial centre.

In fact, for the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_4, \mathfrak{g}_8, \mathfrak{g}_9$ it was already known that every complete left symmetric structure has a non-trivial centre [7,6,11]. So, in this paper we only need to prove this result for the Lie algebra $\mathfrak{g}_7$.

2. Simply transitive actions and left symmetric structures

As we already mentioned we shall prove all the results on the Lie algebra level. Therefore, we need to know more about the relationship between simply transitive affine actions of a Lie group G and complete left symmetric structures on the corresponding Lie algebra $\mathfrak{g}$. (All Lie algebras in this paper will be finite dimensional and over the field $\mathbb{R}$). For details on this relationship, we refer the reader to [8–10,14].

Let $\mathrm{Aff}(\mathbb{R}^n)$ denote the group of invertible affine motions on $\mathbb{R}^n$ and assume that

$$\rho : G \to \mathrm{Aff}\big(\mathbb{R}^n\big) : g \mapsto \begin{pmatrix} \rho_L(g) & \rho_T(g) \\ 0 & 1 \end{pmatrix}$$

determines a simply transitive action of a nilpotent Lie group on $\mathbb{R}^n$. The map $\rho_L : G \to \mathrm{GL}(\mathbb{R}^n) : g \mapsto \rho_L(g)$ (respectively $\rho_T : G \to \mathbb{R}^n : g \mapsto \rho_T(g)$) will be called the linear (respectively translational) part of ρ.

Its differential $d\rho$ induces a Lie algebra morphism

$$d\rho : \mathfrak{g} \to \mathfrak{aff}\big(\mathbb{R}^n\big) : g \mapsto \begin{pmatrix} d\rho_L(g) & d\rho_T(g) \\ 0 & 0 \end{pmatrix}$$

with translational part $d\rho_T$ and linear part $d\rho_L$. It is known that for nilpotent Lie groups G, with corresponding Lie algebra $\mathfrak{g}$, $d\rho_L(\mathfrak{g})$ consists of nilpotent matrices and $d\rho_T:\mathfrak{g}\to\mathbb{R}^n$ is a linear isomorphism. Also the converse is true, any representation $\varphi:\mathfrak{g}\to\mathfrak{aff}(\mathbb{R}^n)$, for which the linear part $\varphi_L(\mathfrak{g})$ consists of nilpotent matrices and the translational part $\varphi_T:\mathfrak{g}\to\mathbb{R}^n$ is bijective, is in fact the differential $d\rho$ of a simply transitive affine action $\rho:G\to\mathrm{Aff}(\mathbb{R}^n)$. Such a φ is called a complete affine structure on $\mathfrak{g}$. Hence there is a one-to-one correspondence between simply transitive affine actions ρ of a nilpotent Lie group G and complete affine structures $d\rho$ on its Lie algebra $\mathfrak{g}$.

Secondly, affine structures on a Lie algebra are related to left symmetric structures on this Lie algebra.

Definition 2.1. Let $\mathfrak{g}$ be a Lie algebra. A left symmetric structure on $\mathfrak{g}$ consists of a bilinear product $\bullet:\mathfrak{g}\times\mathfrak{g}\to\mathfrak{g}$, satisfying

(1) $\forall X,Y\in\mathfrak{g}$: $[X,Y]=X\bullet Y-Y\bullet X$, and
(2) $\forall X,Y,Z\in\mathfrak{g}$: $[X,Y]\bullet Z=X\bullet(Y\bullet Z)-Y\bullet(X\bullet Z)$.

Moreover, a left symmetric structure is said to be complete if and only if $\forall Y\in\mathfrak{g}$, the map $t_Y:\mathfrak{g}\to\mathfrak{g}:X\mapsto X+X\bullet Y$ is bijective.

If $\mathfrak{g}$ is a nilpotent Lie algebra and $\varphi:\mathfrak{g}\to\mathfrak{aff}(\mathbb{R}^n)$ is a complete affine structure, then we can define a left symmetric structure on $\mathfrak{g}$ by

$$\forall X,Y\in\mathfrak{g}:\ X\bullet Y=\varphi_T^{-1}\big(\varphi_L(X)\varphi_T(Y)\big).$$

Moreover, this will be a complete left symmetric structure.

Conversely, assume a given left symmetric structure $\bullet$ on $\mathfrak{g}$. By choosing a basis, we identify $\mathfrak{g}$ with $\mathbb{R}^n$ via its coordinate map $\mathrm{co}:\mathfrak{g}\to\mathbb{R}^n:X\mapsto\mathrm{co}(X)$, the coordinate of X with respect to the chosen basis. An affine representation $\varphi:\mathfrak{g}\to\mathfrak{aff}(\mathbb{R}^n)$ can then be defined by taking

$$\varphi_T=\mathrm{co}\quad\text{and}\quad\forall X\in\mathfrak{g},\ \forall a\in\mathbb{R}^n:\ \varphi_L(X)a=\mathrm{co}\big(X\bullet\mathrm{co}^{-1}(a)\big).$$

Moreover, this affine structure is complete if and only if $\bullet$ is a complete left symmetric structure. This explains the one-to-one correspondence between complete affine structures and complete left symmetric structures on a nilpotent Lie algebra.

We are now interested in the simply transitive affine structures on a nilpotent Lie group G for which the identity element is the only element acting as a pure translation. Such actions exactly correspond to the complete left symmetric structures on the corresponding Lie algebra $\mathfrak{g}$ for which the set $T(\mathfrak{g})=0$, where

$$T(\mathfrak{g})=\{X\in\mathfrak{g}\parallel X\bullet Y=0,\ \forall Y\in\mathfrak{g}\}.$$

Before we can really take off with our investigation, we need to recall some more facts about complete left symmetric structures $\bullet$ on a nilpotent Lie algebra $\mathfrak{g}$.

We define the left multiplication map λ:

$$\lambda:\mathfrak{g}\to\mathfrak{gl}(\mathfrak{g}):X\mapsto\lambda_X,\quad\text{with }\lambda_X:\mathfrak{g}\to\mathfrak{g}:Y\mapsto\lambda_X(Y)=X\bullet Y$$

and the right multiplication map ρ:

$$\rho : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) : X \mapsto \rho_X, \quad \text{with } \rho_X : \mathfrak{g} \to \mathfrak{g} : Y \mapsto \rho_X(Y) = Y \bullet X.$$

From Definition 2.1 it follows that λ is a Lie algebra homomorphism, while ρ need not be. H. Kim showed that λ_X and ρ_X are nilpotent for any $X \in \mathfrak{g}$ [10]. Now we can consider the following two decreasing sequences of Lie subalgebras of $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 = \mathfrak{g} \cdot \mathfrak{g} \supseteq \mathfrak{g}^3 = \mathfrak{g} \cdot \mathfrak{g}^2 \supseteq \cdots \supseteq \mathfrak{g}^{i+1} = \mathfrak{g} \cdot \mathfrak{g}^i \supseteq \cdots$$

and

$$\mathfrak{g} = \mathfrak{g}_1 \supseteq \mathfrak{g}_2 = \mathfrak{g} \cdot \mathfrak{g} \supseteq \mathfrak{g}_3 = \mathfrak{g}_2 \cdot \mathfrak{g} \supseteq \cdots \supseteq \mathfrak{g}_{i+1} = \mathfrak{g}_i \cdot \mathfrak{g} \supseteq \cdots.$$

For the first of these sequences, we have that, for sufficiently large n, $\mathfrak{g}^n = 0$. This follows from the fact that $\lambda : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ is a Lie algebra homomorphism whose image consists of nilpotent endomorphisms. The second sequence also stabilizes after a finite number of steps, but this need not be at zero. Therefore, we define

$$\mathfrak{g}_\infty = \bigcap_{n=1}^{\infty} \mathfrak{g}_n.$$

Now, we still need some notations. Firstly, the centre of $\mathfrak{g}$, considered as a Lie algebra is denoted by

$$Z(\mathfrak{g}) = \bigl\{X \in \mathfrak{g} \parallel [X, Y] = 0,\ \forall Y \in \mathfrak{g}\bigr\} = \{X \in \mathfrak{g} \parallel X \bullet Y = Y \bullet X,\ \forall Y \in \mathfrak{g}\}$$

and the centre of $\mathfrak{g}$, considered as a left symmetric algebra is given by

$$C(\mathfrak{g}) = \{X \in \mathfrak{g} \parallel X \bullet Y = 0 = Y \bullet X,\ \forall Y \in \mathfrak{g}\}.$$

For a complete left symmetric structure $\bullet$ on a nilpotent Lie algebra $\mathfrak{g}$, one has that $T(\mathfrak{g}) = 0 \iff C(\mathfrak{g}) = 0$. This follows from the fact that $T(\mathfrak{g})$ is a (Lie algebra) ideal of $\mathfrak{g}$, which has a non-empty intersection with $Z(\mathfrak{g})$, in case $T(\mathfrak{g}) \neq 0$.
The centralizer of an element $Y \in \mathfrak{g}$ is defined by

$$C_{\mathfrak{g}}(Y) = \bigl\{X \in \mathfrak{g} \parallel [X, Y] = 0\bigr\}.$$

In the following proposition and lemma we recall some results obtained in [10]:

Proposition 2.2. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$. Then*

(1) *each $\mathfrak{g}_i$ is a two-sided ideal of $\mathfrak{g}$ (thus $\mathfrak{g} \bullet \mathfrak{g}_i \subseteq \mathfrak{g}_i$ and $\mathfrak{g}_i \bullet \mathfrak{g} \subseteq \mathfrak{g}_i$).*
(2) $\mathfrak{g}_\infty \bullet \mathfrak{g} = \mathfrak{g}_\infty$.
(3) *$\mathfrak{g}_\infty$ is a proper ideal of $\mathfrak{g}$.*
(4) *if $\mathfrak{g}_\infty \neq 0$, then $\dim(\mathfrak{g}_\infty) \geqslant 3$.*
(5) *if $T(\mathfrak{g}) = 0$, then $\mathfrak{g}_\infty \neq 0$.*

Lemma 2.3. *Let $\bullet$ be a complete left symmetric structure on a nilpotent Lie algebra $\mathfrak{g}$, then there exists a basis $e_1, \ldots, e_r, e_{r+1}, \ldots, e_n$ of $\mathfrak{g}$ such that $\mathfrak{g}_\infty = \operatorname{span}\langle e_1, e_2, \ldots, e_r\rangle$, and with respect to this basis, the matrices of $\lambda(\mathfrak{g})$ are of the form*

$$\begin{pmatrix} A_r & * \\ 0 & B_{n-r} \end{pmatrix}$$

and those of $\rho(\mathfrak{g})$ *are of the form*

$$\begin{pmatrix} C_r & * \\ 0 & D_{n-r} \end{pmatrix},$$

where A_r ($r \times r$ *matrix*), B_{n-r} *and* D_{n-r} ($(n-r) \times (n-r)$*-matrices*) *are simultaneously strict upper triangular.*

Let V be a vector space, equipped with a filtration of subspaces

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V,$$

where the dimension of $V_i/V_{i-1} = k_i$ $(1 \leqslant i \leqslant n)$. We say that a basis $v_1, v_2, \ldots, v_k$ (with $k = k_1 + k_2 + \cdots + k_n$) is compatible with the given filtration, if V_i is spanned by $v_1, v_2, \ldots, v_{k_1+k_2+\cdots+k_i}$, for each i.

To obtain the basis $e_1, e_2, \ldots, e_n$ referred to in Lemma 2.3, one chooses the subbasis $e_1, \ldots, e_r$ for $\mathfrak{g}_\infty$ to be compatible with the filtration of $\mathfrak{g}_\infty$ induced by left multiplication:

$$0 \subset \cdots \subset \mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g}_\infty) \subset \mathfrak{g} \bullet \mathfrak{g}_\infty \subset \mathfrak{g}_\infty,$$

and the rest of the vectors $e_{r+1}, \ldots, e_n$ in such a way that their natural projections $\overline{e_{r+1}}, \ldots, \overline{e_n}$ in $\mathfrak{g}/\mathfrak{g}_\infty$ form a basis which is compatible with the filtration of $\mathfrak{g}/\mathfrak{g}_\infty$ induced by right multiplication:

$$0 \subset \cdots \subset (\mathfrak{g} \bullet \mathfrak{g}) \bullet \mathfrak{g}/\mathfrak{g}_\infty \subset \mathfrak{g} \bullet \mathfrak{g}/\mathfrak{g}_\infty \subset \mathfrak{g}/\mathfrak{g}_\infty.$$

Although there might exist other bases for which Lemma 2.3 holds, we will in this paper always assume that $e_1, e_2, \ldots, e_n$ are chosen in the way mentioned above and we will call such a basis an *adapted basis*. So, for an adapted basis Lemma 2.3 is valid.

Remark 2.4. Note that for an adapted basis $e_1, \ldots, e_n$, $X \bullet e_1 = 0$, for all $X \in \mathfrak{g}$.

Finally, we recall some results obtained in [6]: (the first statement is formulated more generally, so we still need to prove this)

Lemma 2.5. *Let* $\bullet$ *be a complete left symmetric structure on a nilpotent Lie algebra* $\mathfrak{g}$, *then*

(1) $[\mathfrak{g}, \mathfrak{g}] \nsubseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$.
(2) *if* $T(\mathfrak{g}) = 0$, *then* $\mathfrak{g}_\infty \nsubseteq Z(\mathfrak{g})$. *In fact, if* $e_1, \ldots, e_n$ *is an adapted basis, then* $e_1 \notin Z(\mathfrak{g})$.

Proof. Assume that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$. For any $X \in \mathfrak{g}_\infty$ and any $Y \in \mathfrak{g}$, we have that

$$X \bullet Y = Y \bullet X + [X, Y] \implies X \bullet Y \in \mathfrak{g} \bullet \mathfrak{g}_\infty.$$

It follows that $\mathfrak{g}_\infty \bullet \mathfrak{g} \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$. By Lemma 2.3, we know that $\mathfrak{g} \bullet \mathfrak{g}_\infty$ is a proper subspace of $\mathfrak{g}_\infty$, and therefore also $\mathfrak{g}_\infty \bullet \mathfrak{g} \neq \mathfrak{g}_\infty$, which contradicts Proposition 2.2 and proves the first statement. The second statement was already obtained in [6]. $\square$

3. Lie groups not admitting simply transitive affine actions without translations

We have already mentioned that every complete left symmetric structure on one of the Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2, \mathfrak{g}_4, \mathfrak{g}_8$ and $\mathfrak{g}_9$ (see the classification in Theorem 1.1) has a non-trivial centre (see [7,6,11]). So the Auslander conjecture is true for the corresponding Lie groups. In order to obtain the first step of the main result, we still need to prove that the same holds for $\mathfrak{g}_7$, which is the free 3-step nilpotent Lie algebra on 2 generators. For this Lie algebra $\mathfrak{g}_7$, we have that $\dim([\mathfrak{g}_7, \mathfrak{g}_7]) = 3$, $\dim([\mathfrak{g}_7, [\mathfrak{g}_7, \mathfrak{g}_7]]) = 2$ and $[\mathfrak{g}_7, [\mathfrak{g}_7, \mathfrak{g}_7]] = Z(\mathfrak{g}_7)$.

Any two elements v and w in $\mathfrak{g}_7$, for which their canonical projections in $\mathfrak{g}_7/[\mathfrak{g}_7, \mathfrak{g}_7]$ are linear independent generate $\mathfrak{g}_7$ as a Lie algebra and induce a vector space decomposition for $\mathfrak{g}_7$:

$$\mathfrak{g}_7 = V_1 \oplus V_2 \oplus V_3, \tag{1}$$

where V_1 is the vector space generated by the two generators v and w of $\mathfrak{g}_7$, $V_2 = [V_1, V_1] = \langle [v, w] \rangle$ and $V_3 = [V_1, V_2]$. For any such generating v and w, we have that $\dim(V_1) = 2$, $\dim(V_2) = 1$, $\dim(V_3) = 2$ and that $[\mathfrak{g}_7, \mathfrak{g}_7] = V_2 \oplus V_3$, $V_3 = Z(\mathfrak{g}_7)$.

Before we can prove that every complete left symmetric structure on $\mathfrak{g}_7$ has a non-trivial centre, we need some lemmas.

Lemma 3.1. *Let $\bullet$ be a complete left symmetric structure on the free* 3*-step nilpotent Lie algebra* $\mathfrak{g}$ *on* 2 *generators. Then if* $\mathfrak{g}_\infty \neq 0$, $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\infty$.

Proof. Assume that $\mathfrak{g}_\infty \neq 0$. Recall that there is a vector space decomposition (1) of $\mathfrak{g}$ induced by the choice of the two generators. Let $e_1, e_2, \ldots, e_5$ be an adapted basis for $\mathfrak{g}$. Then $e_1 \in \mathfrak{g}_\infty$, but by Lemma 2.5 $e_1 \notin Z(\mathfrak{g}) = V_3$. By choosing a different pair of generators if necessary, we can suppose that $e_1 \in V_1$ or $e_1 \in V_2$. Let us first consider the case $e_1 \in V_1$. Because V_2 is one-dimensional, we have $V_2 = [e_1, V_1]$ and therefore $V_2 \subseteq \mathfrak{g}_\infty$. In the second case $V_2 = \langle e_1 \rangle \subseteq \mathfrak{g}_\infty$. So, in both situations, we find that $V_2 \subseteq \mathfrak{g}_\infty$. Moreover, as $V_3 = [V_1, V_2]$, it follows that V_3 also is a subset of $\mathfrak{g}_\infty$. This proves $V_2 \oplus V_3 = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\infty$. □

Lemma 3.2. *Let $\bullet$ be a complete left symmetric structure on the free* 3*-step nilpotent Lie algebra* $\mathfrak{g}$ *on* 2 *generators, with* $T(\mathfrak{g}) = 0$ *and* $e_1 \notin [\mathfrak{g}, \mathfrak{g}]$*, then*

(1) $\mathfrak{g}_\infty = \langle e_1, [\mathfrak{g}, \mathfrak{g}] \rangle = \mathfrak{g} \bullet \mathfrak{g}$.
(2) $\mathfrak{g} \bullet \mathfrak{g}_\infty = \langle e_1, [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \rangle$ *and is a* 3*-dimensional Abelian Lie algebra.*

Proof. As $e_1 \notin [\mathfrak{g}, \mathfrak{g}]$, we may assume that e_1 is one of the generators of $\mathfrak{g}$ and so for the induced decomposition (1), $e_1 \in V_1$. We know that $[\mathfrak{g}, \mathfrak{g}] = V_2 \oplus V_3$, and $e_1 \notin [\mathfrak{g}, \mathfrak{g}]$ so $\dim(\langle e_1, [\mathfrak{g}, \mathfrak{g}] \rangle) = 4$. By definition of e_1 and Lemma 3.1, we have $e_1 \in \mathfrak{g}_\infty$ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\infty$ and therefore $\langle e_1, [\mathfrak{g}, \mathfrak{g}] \rangle \subseteq \mathfrak{g}_\infty$. On the other hand $\mathfrak{g}_\infty$ is a proper ideal of $\mathfrak{g}$ (Proposition 2.2), so this proves (1).

Now, we will prove that $\mathfrak{g} \bullet \mathfrak{g}_\infty = \langle e_1, V_3 \rangle$ (with $V_3 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$). It is easy to see that this is a 3-dimensional Abelian Lie algebra. We compute

$$\begin{aligned} V_3 = [V_1, V_2] &\subseteq V_1 \bullet V_2 - V_2 \bullet V_1 \\ &\subseteq V_1 \bullet V_2 - [V_1, V_1] \bullet V_1 \\ &\subseteq V_1 \bullet V_2 - V_1 \bullet (V_1 \bullet V_1) \\ &\subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty, \end{aligned}$$

where we used that $V_1 \bullet V_2 \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$, and $V_1 \bullet (V_1 \bullet V_1) \subseteq \mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g}) = \mathfrak{g} \bullet \mathfrak{g}_\infty$. By the definition of e_1 (as the first element of the adapted basis) it follows that $e_1 \in \mathfrak{g} \bullet \mathfrak{g}_\infty$ (note that $\mathfrak{g} \bullet \mathfrak{g}_\infty \neq 0$, since it contains $[V_1, V_2]$). Hence $\langle e_1, V_3 \rangle \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$. We know by Lemma 2.3 that $\mathfrak{g} \bullet \mathfrak{g}_\infty$ is a proper subspace of $\mathfrak{g}_\infty$, so $\mathfrak{g} \bullet \mathfrak{g}_\infty = \langle e_1, V_3 \rangle$, which proves the second statement. □

Remark 3.3. In the proof of the previous lemma we used the notation $[A, B] \subseteq A \bullet B - B \bullet A$, for two subspaces A and B of $\mathfrak{g}$. Of course, $A \bullet B - B \bullet A = A \bullet B + B \bullet A$, but we prefer the notation using the minus sign, because the inclusion follows from the rule $[a, b] = a \bullet b - b \bullet a$ which holds for elements of $\mathfrak{g}$. In the sequel of the paper, we will often follow this principle for the readers comfort.

Lemma 3.4. *Let $\bullet$ be a complete left symmetric structure on the free* 3*-step nilpotent Lie algebra $\mathfrak{g}$ on* 2 *generators, with $T(\mathfrak{g}) = 0$ and $e_1 \in [\mathfrak{g}, \mathfrak{g}]$, then*

(1) $\mathfrak{g}_\infty = [\mathfrak{g}, \mathfrak{g}]$.
(2) $\mathfrak{g}_\infty$ *is a* 3*-dimensional Abelian Lie algebra.*
(3) $\mathfrak{g}_\infty \subsetneq \mathfrak{g} \bullet \mathfrak{g}$ (*thus $\mathfrak{g} \bullet \mathfrak{g}$ is* 4*-dimensional*).
(4) $\mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g}) \subseteq \mathfrak{g}_\infty$.
(5) $\mathfrak{g}_\infty \bullet \mathfrak{g}_\infty = 0$.

Proof. As $e_1 \notin Z(\mathfrak{g})$, we can assume that $e_1 \in V_2$ in a decomposition (1) induced by an appropriate pair of generators. First we prove that $V_3 \subseteq \mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g})$:

$$\begin{aligned} V_3 = [V_1, V_2] &= \big[V_1, [V_1, V_1]\big] \\ &\subseteq V_1 \bullet [V_1, V_1] - [V_1, V_1] \bullet V_1 \\ &\subseteq \mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g}), \end{aligned}$$

where we used the Definition 2.1.

By Lemma 3.1 we know that $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\infty$, so let us now assume that $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}_\infty$, which implies that $\dim(\mathfrak{g}_\infty) = 4$. Hence $\mathfrak{g}_\infty = \mathfrak{g} \bullet \mathfrak{g}$, from which it follows that $V_3 \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$. By definition of e_1, we also have that $V_2 = \langle e_1 \rangle \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$. Therefore $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$, which contradicts Lemma 2.5, and proves (1).

The second statement is an immediate consequence of the first.

For the proof of statement (3), let us assume that $\mathfrak{g}_\infty = \mathfrak{g} \bullet \mathfrak{g}$. Now we can use the same argument as in the proof of statement (1).

We know that $\mathfrak{g} \bullet (\mathfrak{g} \bullet \mathfrak{g}) \subseteq (\mathfrak{g} \bullet \mathfrak{g}) \bullet \mathfrak{g} + [\mathfrak{g}, \mathfrak{g} \bullet \mathfrak{g}]$. By Lemma 3.1, it follows that $[\mathfrak{g}, \mathfrak{g} \bullet \mathfrak{g}] \subseteq \mathfrak{g}_\infty$. On the other hand, because $\dim(\mathfrak{g}_\infty) = 3$, we have $(\mathfrak{g} \bullet \mathfrak{g}) \bullet \mathfrak{g} = \mathfrak{g}_\infty$. This proves (4).

Now we prove that $V_i \bullet V_j = 0$ for $i, j \in \{2, 3\}$. As $V_2 = \langle e_1 \rangle$, it follows that $\mathfrak{g} \bullet V_2 = 0$. Since $\mathfrak{g}_\infty$ is Abelian, we know $V_2 \bullet V_3 = V_3 \bullet V_2 = 0$. We still need to prove $V_3 \bullet V_3 = 0$. Therefore we compute:

$$\begin{aligned} V_3 \bullet V_3 &= [e_1, V_1] \bullet V_3 \\ &\subseteq e_1 \bullet (V_1 \bullet V_3) - V_1 \bullet (e_1 \bullet V_3) \\ &\subseteq [e_1, V_1 \bullet V_3] - V_1 \bullet [e_1, V_3] \\ &\subseteq 0, \end{aligned}$$

where we used that $V_1 \bullet V_3 \subseteq \mathfrak{g}_\infty$ and $\mathfrak{g}_\infty$ is Abelian. $\square$

Now we are ready to prove the following theorem, which will finish the first part of the main result:

Theorem 3.5. *Let $\mathfrak{g}$ be the free 3-step nilpotent Lie algebra on 2 generators. Then for any complete left symmetric structure on $\mathfrak{g}$, we have that $T(\mathfrak{g}) \neq 0$.*

Proof. Assume that $\bullet$ is a complete left symmetric structure, for which $T(\mathfrak{g}) = 0$. In this case we have $\mathfrak{g}_\infty \neq 0$ and $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathfrak{g}_\infty$ (Lemma 3.1). We can distinguish two situations, namely $e_1 \notin [\mathfrak{g}, \mathfrak{g}]$ or $e_1 \in [\mathfrak{g}, \mathfrak{g}]$. In the first situation e_1 can be taken as a generator of $\mathfrak{g}$ and therefore we may assume that $e_1 \in V_1$. For the second situation, $e_1 \in [\mathfrak{g}, \mathfrak{g}] \setminus [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ (by Lemma 2.5) and so there is no loss in generality in assuming that $e_1 \in V_2$.

We will show that both cases lead to a contradiction.

Case 1: $e_1 \in V_1$ ($e_1 \notin [\mathfrak{g}, \mathfrak{g}]$).
We want to show that there exists an element $v_3 \in V_3$ such that $v_3 \bullet \mathfrak{g} = 0$. First let us take an element $v_1 \in V_1$ and $v_2 \in V_2$ such that $V_1 = \langle e_1, v_1 \rangle$ and $v_2 = [e_1, v_1]$ (thus $V_2 = \langle v_2 \rangle$).

We already know that $V_3 \bullet e_1 = 0$. The following computation proves that $V_3 \bullet V_2 = 0$.

$$\begin{aligned} V_3 \bullet V_2 = V_2 \bullet V_3 &= [e_1, V_1] \bullet V_3 \\ &\subseteq e_1 \bullet (V_1 \bullet V_3) - V_1 \bullet (e_1 \bullet V_3) \\ &\subseteq [e_1, V_1 \bullet V_3] - V_1 \bullet [e_1, V_3] \\ &= 0, \end{aligned}$$

where we used that $V_3 = Z(\mathfrak{g})$, $\langle e_1, V_1 \bullet V_3 \rangle \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$ and $\mathfrak{g} \bullet \mathfrak{g}_\infty$ is Abelian (Lemma 3.2). We also have that $V_2 \bullet \langle e_1, V_3 \rangle = V_2 \bullet (\mathfrak{g} \bullet \mathfrak{g}_\infty) = 0$ (Lemma 3.2).

As $V_1 = \langle e_1, v_1 \rangle$, this implies that $V_3 = \langle [e_1, V_2], [v_1, V_2] \rangle$. Now,

$$\begin{aligned} [e_1, V_2] \bullet V_3 &\subseteq [e_1, V_2 \bullet V_3] - V_2 \bullet [e_1, V_3] \\ &= 0, \\ [v_1, V_2] \bullet V_3 &\subseteq v_1 \bullet (V_2 \bullet V_3) - V_2 \bullet (v_1 \bullet V_3) \\ &= 0, \end{aligned}$$

where we used that $V_3 = Z(\mathfrak{g})$ and $V_2 \bullet (\mathfrak{g} \bullet \mathfrak{g}_\infty) = 0$, which proves that $V_3 \bullet V_3 = 0$. Thus we already obtained that $V_3 \bullet \mathfrak{g}_\infty = 0$ (Lemma 3.2). If we can find an element $v_3 \in V_3$ such that $v_3 \bullet v_1 = 0$, then we also have $v_3 \bullet \mathfrak{g} = 0$, which would finish the proof in this case.

Because $\mathfrak{g}_\infty = \mathfrak{g} \bullet \mathfrak{g}$, and $\mathfrak{g} \bullet \mathfrak{g}_\infty = \langle e_1, V_3 \rangle$, there exists an element $b \in \mathbb{R}$ and $l \in \mathfrak{g} \bullet \mathfrak{g}_\infty$ such that $v_1 \bullet v_1 = l + bv_2$. Now, to find this element v_3, we need some computations.

$$\begin{aligned}
v_2 \bullet v_1 &= [e_1, v_1] \bullet v_1 \\
&= [e_1, v_1 \bullet v_1] - v_1 \bullet [e_1, v_1] \\
&= [e_1, v_1 \bullet v_1] - v_1 \bullet v_2 \\
&= [e_1, v_1 \bullet v_1] - v_2 \bullet v_1 - [v_1, v_2] \\
&= \frac{1}{2}[e_1, v_1 \bullet v_1] - \frac{1}{2}[v_1, v_2] \in V_3 \quad (\mathfrak{g} \bullet \mathfrak{g}_\infty \text{ Abelian}),
\end{aligned}$$

$$\begin{aligned}
[e_1, v_2] \bullet v_1 &= [e_1, v_2 \bullet v_1] - v_2 \bullet [e_1, v_1] \\
&= -v_2 \bullet [e_1, v_1] \\
&= -v_2 \bullet v_2,
\end{aligned}$$

$$\begin{aligned}
[v_1, v_2] \bullet v_1 &= v_1 \bullet (v_2 \bullet v_1) - v_2 \bullet (v_1 \bullet v_1) \\
&= \frac{1}{2} v_1 \bullet [e_1, v_1 \bullet v_1] - \frac{1}{2} v_1 \bullet [v_1, v_2] - v_2 \bullet (v_1 \bullet v_1) \\
&= \frac{1}{3} v_1 \bullet [e_1, v_1 \bullet v_1] - \frac{2}{3} v_2 \bullet (v_1 \bullet v_1) \\
&= \frac{b}{3} v_1 \bullet [e_1, v_2] - \frac{2b}{3} v_2 \bullet v_2 \quad (V_2 \bullet (\mathfrak{g} \bullet \mathfrak{g}_\infty) = 0) \\
&= -bv_2 \bullet v_2.
\end{aligned}$$

We now define $v_3 = [v_1 - be_1, v_2]$ $(v_3 \neq 0)$. Then

$$v_3 \bullet v_1 = [v_1 - be_1, v_2] \bullet v_1 = [v_1, v_2] \bullet v_1 - b[e_1, v_2] \bullet v_1 = 0.$$

So we found a $v_3 \neq 0$ with $v_3 \bullet \mathfrak{g} = 0$, which contradicts $T(\mathfrak{g}) = 0$.

Case 2: $e_1 \in V_2$ $(e_1 \in [\mathfrak{g}, \mathfrak{g}])$.
We can choose the generators $v_0, v_1 \in V_1$ of $\mathfrak{g}$ (so $V_1 = \langle v_0, v_1 \rangle$) in such a way that v_0 belongs to $\mathfrak{g} \bullet \mathfrak{g}$. Then $\mathfrak{g} \bullet \mathfrak{g} = \langle v_0, \mathfrak{g}_\infty \rangle$ (Lemma 3.4), hence there exists an adapted basis with $\mathfrak{g}_\infty = \langle e_1, e_2, e_3 \rangle$ and $v_0 = e_4$. Therefore $\mathfrak{g} \bullet v_0 \subseteq \mathfrak{g}_\infty$.

We first show that the product $v_1 \bullet v_1 \notin \mathfrak{g}_\infty$. We know:

$$\begin{aligned}
&\mathfrak{g} \bullet \mathfrak{g}_\infty \subseteq \mathfrak{g}_\infty, \\
&\mathfrak{g} \bullet v_0 \subseteq \mathfrak{g}_\infty, \\
&\mathfrak{g}_\infty \bullet \ v_1 \subseteq \mathfrak{g}_\infty, \\
&v_0 \bullet v_1 = v_1 \bullet v_0 + [v_0, v_1] \in \mathfrak{g}_\infty \quad \text{(Lemma 3.1)}.
\end{aligned}$$

Assuming that $v_1 \bullet v_1 \in \mathfrak{g}_\infty$ leads to $\mathfrak{g} \bullet \mathfrak{g} \subseteq \mathfrak{g}_\infty$, which contradicts Lemma 3.4. Thus there exists an element $l \in \mathfrak{g}_\infty, b \in \mathbb{R}_0$ such that $v_1 \bullet v_1 = l + bv_0$.

As a second step we prove that $[e_1, v_0] \bullet v_1 \neq 0$. Therefore we need following computation

$$[e_1, v_i] \bullet v_j = [e_1, v_i \bullet v_j] - v_i \bullet [e_1, v_j], \quad i, j \in \{0, 1\}. \tag{2}$$

We know $v_0 \bullet v_0 \in \mathfrak{g}_\infty$ and by Lemma 3.4 $\mathfrak{g}_\infty$ is Abelian, so $[e_1, v_0] \bullet v_0 = 0$. Let us now assume that $[e_1, v_0] \bullet v_1 = 0$. It follows that $[e_1, v_0] \bullet V_1 = 0$. But, by the same lemma we also know that $[e_1, v_0] \bullet \mathfrak{g}_\infty = 0$. Hence $[e_1, v_0] \in T(\mathfrak{g})$, which is a contradiction.

As last step we now compute $([e_1, v_1] \bullet v_1) \bullet v_1$ in two different ways.

$$\begin{aligned}
&[e_1, v_1] \bullet v_0 = -[e_1, v_0] \bullet v_1 \quad \text{(follows from (2))}, \\
&[e_1, v_1] \bullet v_1 = \frac{1}{2}[e_1, v_1 \bullet v_1] = \frac{b}{2}[e_1, v_0] \in V_3, \\
&\big([e_1, v_1] \bullet v_1\big) \bullet v_1 = \frac{b}{2}[e_1, v_0] \bullet v_1, \\
&\big([e_1, v_1] \bullet v_1\big) \bullet v_1 = v_1 \bullet \big([e_1, v_1] \bullet v_1\big) \\
&\qquad = \big[v_1, [e_1, v_1]\big] \bullet v_1 + [e_1, v_1] \bullet (v_1 \bullet v_1) \\
&\qquad = -b[e_1, v_0] \bullet v_1 \quad \text{(Lemma 3.4)}.
\end{aligned}$$

So, we find

$$\frac{b}{2}[e_1, v_0] \bullet v_1 = -b[e_1, v_0] \bullet v_1.$$

This is a contradiction with $b \neq 0$ and $[e_1, v_0] \bullet v_1 \neq 0$. $\square$

Remark 3.6. Theorem 3.5 tells us that the Auslander conjecture is true for the free 3-step nilpotent Lie algebra on 2 generators. As the Auslander conjecture is also valid for Abelian Lie algebras and the 3-dimensional Heisenberg Lie algebra, one might think that Auslander's conjecture holds for all free nilpotent Lie algebras. However, in a forthcoming paper [5] we construct counter examples on any free 2-step nilpotent Lie algebra with an odd number of generators $k \geqslant 5$. On the other hand, we also prove that Auslander's conjecture does hold for the free 2-step nilpotent Lie algebra on 3 generators.

4. Two-step nilpotent Lie groups admitting simply transitive affine actions without translations

If we take a look at the classification of all nilpotent 5-dimensional Lie algebras, there is only one 2-step nilpotent Lie algebra that we have not discussed already, namely $\mathfrak{g} = \mathfrak{g}_5$. In this section we want to show that there exist complete left symmetric structures on $\mathfrak{g}_5$ which have a trivial centre (or $T(\mathfrak{g}_5) = 0$). We will also give a classification of these complete left symmetric structures.

First we introduce Lie algebras with Abelian centralizers:

Definition 4.1. We say that a Lie algebra $\mathfrak{g}$ has *Abelian centralizers* if $\forall X \in \mathfrak{g}$: $C_\mathfrak{g}(X) = \mathfrak{g}$ or $C_\mathfrak{g}(X)$ is Abelian.

It is easy to see that the Lie algebra $\mathfrak{g}_5$ satisfies these conditions. Also $\mathfrak{g}_3$ is such a Lie algebra.

Now, we can give some results about these Lie algebras.

Lemma 4.2. *Let* $\mathfrak{g}$ *be a finite-dimensional Lie algebra, then*

$$\forall X \in \mathfrak{g}\colon\ \operatorname{codim}\bigl(C_{\mathfrak{g}}(X)\bigr) = \dim\bigl([\mathfrak{g}, X]\bigr).$$

Proof. This is a corollary of the Rank–Nullity Theorem for linear transformations on

$$[\,\cdot\,, X] : \mathfrak{g} \to \mathfrak{g} : Y \mapsto [Y, X]. \qquad \square$$

Lemma 4.3. *Let* $\bullet$ *be a complete left symmetric structure on a nilpotent Lie algebra* $\mathfrak{g}$*. If* $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) = 1$*, then* $C_{\mathfrak{g}}(e_1)$ *is a right ideal of* $\mathfrak{g}$*.*

Proof. If $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) = 1$, there exists an element $X \in \mathfrak{g}$ such that $\mathfrak{g} = \langle X\rangle \oplus C_{\mathfrak{g}}(e_1)$ (vector space decomposition). Firstly we prove that $C_{\mathfrak{g}}(e_1) \bullet C_{\mathfrak{g}}(e_1) \subseteq C_{\mathfrak{g}}(e_1)$. Take $A \in C_{\mathfrak{g}}(e_1)$ and $B \in C_{\mathfrak{g}}(e_1)$, then

$$e_1 \bullet (A \bullet B) = [e_1, A] \bullet B + A \bullet (e_1 \bullet B) = 0,$$

thus $A \bullet B \in C_{\mathfrak{g}}(e_1)$. Now we still need to prove $C_{\mathfrak{g}}(e_1) \bullet X \subseteq C_{\mathfrak{g}}(e_1)$. Take $A \in C_{\mathfrak{g}}(e_1)$, then there exists an element $\alpha \in \mathbb{R}$ and an element $B_1 \in C_{\mathfrak{g}}(e_1)$ such that $A \bullet X = \alpha X + B_1$. Multiplying on the left again, by A, we find an element $B_2 \in C_{\mathfrak{g}}(e_1)$ such that $\lambda_A(A \bullet X) = A \bullet (A \bullet X) = \alpha^2 X + B_2$. Continuing this way, we have that $\lambda_A^n(X) = \alpha^n X + B_n$, for some $B_n \in C_{\mathfrak{g}}(e_1)$. By the nilpotency of λ_A, there exists an $n \in \mathbb{N}$ for which $A^n \bullet X = 0$. This means that $\alpha = 0$. Hence $A \bullet X \in C_{\mathfrak{g}}(e_1)$, which proves the lemma. $\square$

Lemma 4.4. *Let* $\bullet$ *be a complete left symmetric structure on a nilpotent Lie algebra* $\mathfrak{g}$ *with Abelian centralizers, such that* $T(\mathfrak{g}) = 0$*. Then* $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) \geqslant 2$.

Proof. Assume $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) < 2$. As $e_1 \notin Z(\mathfrak{g})$ (Lemma 2.5), the only possibility is $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) = 1$. By Lemma 4.3 we know that $C_{\mathfrak{g}}(e_1)$ is a right ideal of $\mathfrak{g}$, and $\exists X \in \mathfrak{g}\colon\ \mathfrak{g} = \langle X\rangle \oplus C_{\mathfrak{g}}(e_1)$. Now we define T_1:

$$T_1 = \bigl\{t \in C_{\mathfrak{g}}(e_1) \parallel t \bullet A = 0,\ \forall A \in C_{\mathfrak{g}}(e_1)\bigr\}.$$

By definition of e_1, we know that $e_1 \in T_1$.

Let t be an element of T_1 and $A \in C_{\mathfrak{g}}(e_1)$, then

$$\begin{aligned}(t \bullet X) \bullet A &= A \bullet (t \bullet X)\\ &= [A, t] \bullet X + t \bullet (A \bullet X)\\ &= 0,\end{aligned}$$

where we used Lemma 4.3 and the fact that $C_{\mathfrak{g}}(e_1)$ is Abelian. Hence $t \bullet X \in T_1$ or $T_1 \bullet X \subseteq T_1$. Therefore we can define following nilpotent transformation

$$\rho_X|_{T_1} : T_1 \to T_1 : t \mapsto t \bullet X.$$

It follows that there exists an element $0 \neq t \in T_1$ such that $t \bullet X = 0$, and by definition of T_1, we have also $t \bullet C_{\mathfrak{g}}(e_1) = 0$. Hence $t \bullet \mathfrak{g} = 0$. This is a contradiction with $T(\mathfrak{g}) = 0$. $\square$

The following lemma is only formulated for Lie algebras isomorphic to $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}^m$, for some m. The centre of $\mathfrak{g}$, $Z(\mathfrak{g})$, has codimension 3. This is also a Lie algebra with Abelian centralizers, so we can use previous lemma.

Lemma 4.5. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}^m$. Then $\mathfrak{g}_\infty$ is Abelian.*

Proof. If $\mathfrak{g}_\infty = 0$, there is nothing to show, so we only have to prove the lemma for $\mathfrak{g}_\infty \neq 0$. $\mathfrak{g} = \mathfrak{g}_5 \oplus \mathbb{R}^m$ is a Lie algebra with Abelian centralizers, hence by Lemma 4.4, we know that $\operatorname{codim}(C_{\mathfrak{g}}(e_1)) = 2$. Let us assume that $\mathfrak{g}_\infty$ is not Abelian, then there exists an element e_i $(i \in \{1, \dots, r\})$ in the adapted basis for $\mathfrak{g}$ such that $e_1 \bullet e_i \neq 0$. This is possible because otherwise we should have $\mathfrak{g}_\infty \subseteq C_{\mathfrak{g}}(e_1)$, but we know that $C_{\mathfrak{g}}(e_1)$ is Abelian. Now, define $Y_1 = e_i$ with $i = \min\{k \parallel e_1 \bullet e_k \neq 0\}$. We also know that $[\mathfrak{g}_\infty, \mathfrak{g}_\infty]$ is one-dimensional, since $[\mathfrak{g}_\infty, \mathfrak{g}_\infty] \subseteq \mathfrak{g} \bullet \mathfrak{g}_\infty$ and because of Lemma 2.5. Hence we can find an adapted basis $\langle e_1, \dots, e_r \rangle$ for $\mathfrak{g}_\infty$ such that

$$\begin{aligned} &e_1 \bullet e_k = 0, \quad 1 \leqslant k \leqslant r, k \neq i, \\ &e_1 \bullet e_i \neq 0, \quad e_1 \bullet e_i \in \{e_1, \dots, e_{i-1}\}. \end{aligned}$$

Because $[e_1, \mathfrak{g}] = e_1 \bullet \mathfrak{g}$ is two-dimensional, there exists an element e_{r+l} of the adapted basis for $\mathfrak{g}$ such that $e_1 \bullet e_{r+l} \notin [\mathfrak{g}_\infty, \mathfrak{g}_\infty]$. Define $Y_2 = e_{r+l}$ with $l = \min\{k \parallel e_1 \bullet e_{r+k} \notin [\mathfrak{g}_\infty, \mathfrak{g}_\infty]\}$. Now we can find an adapted basis $\langle e_1, \dots, e_r, \dots, e_n \rangle$ for $\mathfrak{g}$ such that

$$\begin{aligned} &e_1 \bullet e_k = 0, && 1 \leqslant k \leqslant n, k \notin \{i, r+l\}, \\ &e_1 \bullet e_i \neq 0, && e_1 \bullet e_i \in [\mathfrak{g}_\infty, \mathfrak{g}_\infty], \\ &e_1 \bullet e_{r+l} \neq 0, && e_1 \bullet e_{r+l} \notin [\mathfrak{g}_\infty, \mathfrak{g}_\infty]. \end{aligned}$$

Define $Z_1 = e_1 \bullet Y_1 \in \mathfrak{g}_\infty \bullet \mathfrak{g}_\infty$ and $Z_2 = e_1 \bullet e_{r+l} = e_1 \bullet Y_2$ $([\mathfrak{g}, \mathfrak{g}] = \langle Z_1, Z_2 \rangle)$. Hence $Z_2 \in \mathfrak{g}_\infty$, but $Z_2 \notin \mathfrak{g} \bullet \mathfrak{g}_\infty$ (Lemma 2.5). Therefore we can take $e_r = Z_2$. (Remark: it is not possible that $e_r = Y_1$, otherwise Y_1 would be a linear combination of $e_1, \dots, e_{r-1}, Z_2$ and that implies $e_1 \bullet Y_1 = 0$!)

We will now prove that $\mathfrak{g} \bullet \mathfrak{g}_\infty \subseteq \langle e_1, e_2, \dots, e_{r-2} \rangle = \mathfrak{g}'$.
First of all, by definition of the adapted basis for $\mathfrak{g}$, it is enough to prove that $\mathfrak{g} \bullet e_r = \mathfrak{g} \bullet Z_2 \subseteq \mathfrak{g}'$. Let us compute:

$$\begin{aligned} e_k \bullet e_r &= e_r \bullet e_k \in \mathfrak{g}', \quad 1 \leqslant k < r, \\ e_r \bullet e_r &= [e_1, Y_2] \bullet e_r \\ &= e_1 \bullet (Y_2 \bullet Z_2) - Y_2 \bullet [e_1, Z_2] \\ &= e_1 \bullet (Y_2 \bullet Z_2) \in \mathfrak{g}', \\ e_{r+l} \bullet e_r &= Y_2 \bullet Z_2 = Z_2 \bullet Y_2 = [e_1, Y_2] \bullet Y_2 \\ &= \frac{1}{2} e_1 \bullet (Y_2 \bullet Y_2) \\ &\in e_1 \bullet \langle e_1, \dots, e_{r+l-1} \rangle \\ &\in \langle e_1 \bullet e_i \rangle = \langle e_1 \bullet Y_1 \rangle \\ &\in e_1 \bullet \langle e_1, \dots, e_{r-1} \rangle \in \mathfrak{g}'. \end{aligned}$$

We still need to show that $e_m \bullet e_r \in \mathfrak{g}'$, $r < m \leqslant n, m \neq r + l$.

$$\begin{aligned}
e_m \bullet e_r &= e_m \bullet (e_1 \bullet Y_2) = [e_m, e_1] \bullet Y_2 + e_1 \bullet (e_m \bullet Y_2)\\
&= e_1 \bullet (e_m \bullet Y_2)\\
&\in e_1 \bullet \langle e_1, \dots, e_{r+l-1}\rangle \in \mathfrak{g}'.
\end{aligned}$$

This shows that $\mathfrak{g} \bullet \mathfrak{g}_\infty \subseteq \langle e_1, \dots, e_{r-2}\rangle$ or the codimension of $\mathfrak{g} \bullet \mathfrak{g}_\infty$ in $\mathfrak{g}_\infty$ is 2. Therefore we can write

$$\mathfrak{g}_\infty = \mathfrak{g}_\infty \bullet \mathfrak{g} = \mathfrak{g} \bullet \mathfrak{g}_\infty \oplus [\mathfrak{g}, \mathfrak{g}].$$

Hence $Z_1 \notin \mathfrak{g} \bullet \mathfrak{g}_\infty$. This is a contradiction with the definition of $Z_1 = e_1 \bullet Y_1 \in \mathfrak{g} \bullet \mathfrak{g}_\infty$. □

Finally we concentrate on the situation $\mathfrak{g} = \mathfrak{g}_5$. As said before, $\mathfrak{g}_5$ is a 5-dimensional, 2-step nilpotent Lie algebra with Abelian centralizers. $[\mathfrak{g}_5, \mathfrak{g}_5] = Z(\mathfrak{g}_5)$ is two-dimensional. For $\mathfrak{g}_5$ we shall prove that there exist complete left symmetric structures with trivial centre. For all of these complete left symmetric structures, we have that $\mathfrak{g}_{5,\infty} = \mathfrak{g}_5 \bullet \mathfrak{g}_5$. Before we can actually prove this, we need some lemmas.

Lemma 4.6. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_5$, with $T(\mathfrak{g}_5) = 0$. Then*

(1) $\mathfrak{g}_{5,\infty} = \langle e_1, [\mathfrak{g}_5, \mathfrak{g}_5]\rangle$.
(2) $\mathfrak{g}_5 \bullet (\mathfrak{g}_5 \bullet \mathfrak{g}_5) \subseteq \mathfrak{g}_{5,\infty}$.
(3) $\mathfrak{g}_{5,\infty} \bullet \mathfrak{g}_{5,\infty} = 0$.

Proof. As $\mathfrak{g}_{5,\infty} \neq 0$ (because $T(\mathfrak{g}_5) = 0$), $e_1 \in \mathfrak{g}_{5,\infty} \setminus Z(\mathfrak{g}_5)$. Also $[e_1, \mathfrak{g}_5] \subseteq \mathfrak{g}_{5,\infty}$. By Lemma 4.4 $[e_1, \mathfrak{g}_5] = [\mathfrak{g}_5, \mathfrak{g}_5]$ is two-dimensional. Hence $\langle e_1, [\mathfrak{g}_5, \mathfrak{g}_5]\rangle \subseteq \mathfrak{g}_{5,\infty}$. On the other hand, by Lemma 4.5, $\mathfrak{g}_{5,\infty}$ has to be Abelian, so the dimension of $\mathfrak{g}_{5,\infty}$ has to be 3. Therefore $\mathfrak{g}_{5,\infty} = \langle e_1, [\mathfrak{g}_5, \mathfrak{g}_5]\rangle$.

We know that $\mathfrak{g}_5 \bullet (\mathfrak{g}_5 \bullet \mathfrak{g}_5) \subseteq (\mathfrak{g}_5 \bullet \mathfrak{g}_5) \bullet \mathfrak{g}_5 + [\mathfrak{g}_5, \mathfrak{g}_5 \bullet \mathfrak{g}_5]$. Because $\dim(\mathfrak{g}_{5,\infty}) = 3$, it follows that $(\mathfrak{g}_5 \bullet \mathfrak{g}_5) \bullet \mathfrak{g}_5 = \mathfrak{g}_{5,\infty}$ and by statement (1) we also know that $[\mathfrak{g}_5, \mathfrak{g}_5 \bullet \mathfrak{g}_5] \subseteq \mathfrak{g}_{5,\infty}$. This proves (2).

As $\mathfrak{g}_{5,\infty}$ is Abelian, we have $e_1 \bullet \mathfrak{g}_{5,\infty} = \mathfrak{g}_{5,\infty} \bullet e_1 = 0$. Now we compute

$$[\mathfrak{g}_5, \mathfrak{g}_5] \bullet \mathfrak{g}_{5,\infty} = [e_1, \mathfrak{g}_5] \bullet \mathfrak{g}_{5,\infty} \subseteq [e_1, \mathfrak{g}_5 \bullet \mathfrak{g}_{5,\infty}] - \mathfrak{g}_5 \bullet [e_1, \mathfrak{g}_{5,\infty}] = 0.$$

This proves statement (3). □

Lemma 4.7. *Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_5$, with $\mathfrak{g}_{5,\infty} \neq \mathfrak{g}_5 \bullet \mathfrak{g}_5$, then $T(\mathfrak{g}_5) \neq 0$.*

Proof. Assume that $T(\mathfrak{g}_5) = 0$. In this case $\mathfrak{g}_{5,\infty} = \langle e_1, [\mathfrak{g}_5, \mathfrak{g}_5]\rangle$ is an Abelian, three-dimensional Lie subalgebra of $\mathfrak{g}_5$ (Lemma 4.6). Hence $\mathfrak{g}_5 \bullet \mathfrak{g}_5$ is four-dimensional. Therefore there exists an element $Y_1 \in \mathfrak{g}_5 \setminus \mathfrak{g}_{5,\infty}$, such that $\mathfrak{g}_5 \bullet \mathfrak{g}_5 = \langle \mathfrak{g}_{5,\infty}, Y_1\rangle$. Take $Y_2 \in \mathfrak{g} \setminus \mathfrak{g}_5 \bullet \mathfrak{g}_5$. Define $Z_1 = [e_1, Y_1]$ and $Z_2 = [e_1, Y_2]$. This is possible because $\mathrm{codim}(C_{\mathfrak{g}_5}(e_1)) = 2$ (Lemma 4.4). Hence there exists an element $a \in \mathbb{R}$ and an element $l \in \mathfrak{g}_{5,\infty}$ such that $Y_2 \bullet Y_2 = l + aY_1$. Now, we can also find an adapted basis for $\mathfrak{g}_5$ such that $\mathfrak{g}_{5,\infty} = \langle e_1, e_2, e_3\rangle$, $Y_1 = e_4$ and $Y_2 = e_5$. Therefore $\mathfrak{g}_5 \bullet Y_1 \subseteq \mathfrak{g}_{5,\infty}$.

Firstly we will show that $Y_2 \bullet Y_2 \notin \mathfrak{g}_{5,\infty}$ (or $a \neq 0$). Let us assume that $Y_2 \bullet Y_2 \in \mathfrak{g}_{5,\infty}$. We already know that:

$$\begin{aligned}
&\mathfrak{g}_5 \bullet \mathfrak{g}_{5,\infty} \subseteq \mathfrak{g}_{5,\infty},\\
&\mathfrak{g}_5 \bullet Y_1 \subseteq \mathfrak{g}_{5,\infty},\\
&\mathfrak{g}_{5,\infty} \bullet Y_2 \subseteq \mathfrak{g}_{5,\infty},\\
&Y_1 \bullet Y_2 = Y_2 \bullet Y_1 + [Y_1, Y_2] \in \mathfrak{g}_{5,\infty} \quad \text{(Lemma 4.6).}
\end{aligned}$$

Hence $\mathfrak{g}_5 \bullet \mathfrak{g}_5 \subseteq \mathfrak{g}_{5,\infty}$. This is a contradiction with $\mathfrak{g}_{5,\infty} \neq \mathfrak{g}_5 \bullet \mathfrak{g}_5$.

Secondly we compute $Z_i \bullet Y_j$ $(i, j \in \{1, 2\})$.

$$\begin{aligned}
Z_1 \bullet Y_j &= [e_1, Y_1] \bullet Y_j = [e_1, Y_1 \bullet Y_j] - Y_1 \bullet [e_1, Y_j]\\
&= -Z_j \bullet Y_1 \quad (Y_1 \bullet Y_j \in \mathfrak{g}_{5,\infty}),\\
Z_2 \bullet Y_2 &= [e_1, Y_2] \bullet Y_2 = [e_1, Y_2 \bullet Y_2] - Y_2 \bullet [e_1 \bullet Y_2]\\
&= \frac{1}{2}[e_1, Y_2 \bullet Y_2] = \frac{a}{2} Z_1,
\end{aligned}$$

where we used Lemma 4.5. As a corollary we find $Z_1 \bullet Y_1 = 0$. Therefore $Z_1 \bullet Y_2 = -Z_2 \bullet Y_1 \neq 0$ (otherwise $Z_1 \in T(\mathfrak{g}_5)$).

In the last step we compute $Y_2 \bullet (Z_2 \bullet Y_2)$ in two different ways:

$$\begin{aligned}
Y_2 \bullet (Z_2 \bullet Y_2) &= \frac{a}{2} Z_1 \bullet Y_2,\\
Y_2 \bullet (Z_2 \bullet Y_2) &= [Y_2, Z_2] \bullet Y_2 + Z_2 \bullet (Y_2 \bullet Y_2)\\
&= a Z_2 \bullet Y_1 = -a Z_1 \bullet Y_2.
\end{aligned}$$

We have $Z_1 \bullet Y_2 \neq 0$ and $a \neq 0$, so this leads to a contradiction. □

We are still looking for an answer to the question: "Which complete left symmetric structures on $\mathfrak{g}_5$ have a trivial centre". So let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_5$, with $T(\mathfrak{g}_5) = 0$. Then we already know the following results:

- $\mathfrak{g}_{5,\infty} = \langle e_1, [\mathfrak{g}_5, \mathfrak{g}_5] \rangle$, Abelian, three-dimensional Lie algebra (Lemma 4.6),
- $\mathfrak{g}_{5,\infty} \bullet \mathfrak{g}_{5,\infty} = 0$ (Lemma 4.6),
- $\mathfrak{g}_{5,\infty} = \mathfrak{g}_5 \bullet \mathfrak{g}_5$ (Lemma 4.7).

Take $Y_1, Y_2 \in \mathfrak{g}_5 \setminus \mathfrak{g}_{5,\infty}$ such that $[Y_1, Y_2] = 0$. Define $Z_1 = [e_1, Y_1]$ and $Z_2 = [e_1, Y_2]$. Now we compute

$$Z_i \bullet Y_j = [e_1, Y_i] \bullet Y_j = [e_1, Y_i \bullet Y_j] - Y_i \bullet [e_1, Y_j] = -Z_j \bullet Y_i,$$

where we used $Y_i \bullet Y_j \in \mathfrak{g}_{5,\infty}$ and Lemma 4.5. Therefore $\mathfrak{g}_5 \bullet \mathfrak{g}_{5,\infty} = \langle Z_1 \bullet Y_2 \rangle \neq 0$. By definition of e_1, we find $Z_1 \bullet Y_2 = \beta e_1$ $(\beta \in \mathbb{R}_0)$. The only products of elements of the basis which are not completely known, are $Y_i \bullet Y_j$ $(i, j \in \{1, 2\})$. On the other hand we do know that they belong to $\mathfrak{g}_{5,\infty}$, so we can define parameters $a, b, c, d, e, f, g, h, k \in \mathbb{R}$ such that

$$Y_1 \bullet Y_1 = ae_1 + bZ_1 + cZ_2,$$
$$Y_1 \bullet Y_2 = de_1 + eZ_1 + fZ_2 = Y_2 \bullet Y_1,$$
$$Y_2 \bullet Y_2 = ge_1 + hZ_1 + kZ_2.$$

Checking whether this product $\bullet$ defines a complete left symmetric structure with trivial centre, yields two more conditions: $f = -b$ and $k = -e$. So, we find following conclusion:

Let $\bullet$ be a complete left symmetric structure on $\mathfrak{g}_5$, with trivial centre. We can choose an adapted basis $\langle X, Z_1, Z_2, Y_1, Y_2\rangle$ with $[X, Y_1] = Z_1$ and $[X, Y_2] = Z_2$ (the other brackets are zero) such that the product $\bullet$ is given by

$$\begin{array}{lll}
X \bullet X = 0, & Z_1 \bullet X = 0, & Z_2 \bullet X = 0,\\
X \bullet Z_1 = 0, & Z_1 \bullet Z_1 = 0, & Z_2 \bullet Z_1 = 0,\\
X \bullet Z_2 = 0, & Z_1 \bullet Z_2 = 0, & Z_2 \bullet Z_2 = 0,\\
X \bullet Y_1 = Z_1, & Z_1 \bullet Y_1 = 0, & Z_2 \bullet Y_1 = -\beta X,\\
X \bullet Y_2 = Z_2, & Z_1 \bullet Y_2 = \beta X, & Z_2 \bullet Y_2 = 0,
\end{array}$$
$$\begin{array}{ll}
Y_1 \bullet X = 0, & Y_2 \bullet X = 0,\\
Y_1 \bullet Z_1 = 0, & Y_2 \bullet Z_1 = \beta X,\\
Y_1 \bullet Z_2 = -\beta X, & Y_2 \bullet Z_2 = 0,\\
Y_1 \bullet Y_1 = aX + bZ_1 + cZ_2, & Y_2 \bullet Y_1 = dX + eZ_1 - bZ_2,\\
Y_1 \bullet Y_2 = dX + eZ_1 - bZ_2, & Y_2 \bullet Y_2 = gX + hZ_1 - eZ_2,
\end{array} \tag{3}$$

where $\beta \in \mathbb{R}_0$ and $a, b, c, d, e, g, h \in \mathbb{R}$.

From this point onwards, simple computations allow you to classify these left symmetric structures up to isomorphism. (The interested reader can receive details upon easy request.) The conclusion is as follows:

Theorem 4.8. *Up to isomorphism there are* 5 *classes of complete left symmetric structures with trivial centre on the Lie algebra* $\mathfrak{g}_5$. *These are determined by the following parameter settings in the products* (3) *above*:

β	a	d	g	e	b	c	h
1	0	0	0	0	0	0	0
1	0	0	0	0	0	1	0
1	0	0	0	0	0	1	1
1	0	0	0	0	1	0	0
1	0	0	0	0	1	0	−1

For each of these structures $\bullet$ *we have that* $\mathfrak{g}_5 \bullet \mathfrak{g}_5 = \mathfrak{g}_{5,\infty}$ *is* 3-*dimensional.*

5. Three-step nilpotent Lie groups admitting simply transitive affine actions without translations

There are three non-isomorphic 3-step nilpotent Lie algebras of dimension 5, namely $\mathfrak{g}_3, \mathfrak{g}_6, \mathfrak{g}_7$. $\mathfrak{g}_7$ is the free 3-step nilpotent Lie algebra on two generators, and we have already proven that every complete left symmetric structure on $\mathfrak{g}_7$ has a non-trivial centre. We still have to find a classification of the complete left symmetric structures with trivial centre on $\mathfrak{g}_3$ and $\mathfrak{g}_6$. We will not give the entire proof for these Lie algebras, because there are too many computations involved and in fact these computations are very similar to the previous case (i.e. $\mathfrak{g}_5$).

5.1. *Centerless left symmetric structures on* $\mathfrak{g}_3$

Let us first consider $\mathfrak{g}_3$. This is again a Lie algebra with Abelian centralizers. We will prove that if $\bullet$ is a complete left symmetric structure with trivial centre, then $\mathfrak{g}_{3,\infty} = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3]\rangle$ is isomorphic to $\mathfrak{h}_1$, the three-dimensional Heisenberg Lie algebra.

Lemma 5.1. *Let* $\bullet$ *be a complete left symmetric structure on* $\mathfrak{g}_3$*, with* $T(\mathfrak{g}_3) = 0$ *and let* $R \in Z(\mathfrak{g}_3) \setminus [\mathfrak{g}_3, \mathfrak{g}_3]$*. Then* $\mathfrak{g}_3 \bullet \mathfrak{g}_3 \subseteq \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], R\rangle$.

Proof. By Lemma 4.4 we know that $\operatorname{codim}(C_{\mathfrak{g}_3}(e_1)) = 2$ ($\operatorname{codim}(C_{\mathfrak{g}_3}(X)) > 2$ is impossible in $\mathfrak{g}_3, \forall X \in \mathfrak{g}_3$). Take $X \notin \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], R\rangle$. Define $X_3 = [e_1, X]$ and $X_4 = [e_1, X_3]$. Hence $\langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3]\rangle \subseteq \mathfrak{g}_{3,\infty}$. Let us assume $X \in \mathfrak{g}_3 \bullet \mathfrak{g}_3$. Now we have two possibilities: either $X \in \mathfrak{g}_{3,\infty}$ or $X \notin \mathfrak{g}_{3,\infty}$.

Case 1: $X \in \mathfrak{g}_{3,\infty}$. We have $\mathfrak{g}_{3,\infty} = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], X\rangle = \mathfrak{g}_3 \bullet \mathfrak{g}_3$. Hence $X_3 = e_1 \bullet X \in \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$ and $X_4 = e_1 \bullet X_3 \in \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$. Therefore $\langle X_3, X_4\rangle = [\mathfrak{g}_3, \mathfrak{g}_3] \subseteq \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$. This is a contradiction by Lemma 2.5.

Case 2: $X \notin \mathfrak{g}_{3,\infty}$. Now we have $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], X\rangle$ and $\mathfrak{g}_{3,\infty} = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3]\rangle$ or $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle X\rangle \oplus \mathfrak{g}_{3,\infty}$ and $\mathfrak{g}_3 = \langle R\rangle \oplus \mathfrak{g}_3 \bullet \mathfrak{g}_3$. Hence we can find an adapted basis for $\mathfrak{g}_3$ with $\mathfrak{g}_{3,\infty} = \langle e_1, e_2, e_3\rangle$, $X = e_4$ and $R = e_5$. Therefore $\mathfrak{g}_3 \bullet X \subseteq \mathfrak{g}_{3,\infty}$. We compute $\mathfrak{g}_3 \bullet \mathfrak{g}_3$

$$\begin{aligned}\mathfrak{g}_3 \bullet \mathfrak{g}_3 &= \langle \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}, \mathfrak{g}_3 \bullet X, \mathfrak{g}_3 \bullet R\rangle\\ &= \langle \mathfrak{g}_{3,\infty}, R \bullet R\rangle.\end{aligned}$$

So there exist elements $\alpha \in \mathbb{R}_0$, $l \in \mathfrak{g}_{3,\infty}$ such that $R \bullet R = \alpha X + l$. The computation

$$\begin{aligned}[e_1, R] \bullet R = 0 \quad & (R \in Z(\mathfrak{g}_3))\\ &= [e_1, R \bullet R] - R \bullet [e_1, R]\\ &= \beta X_4 + \alpha X_3 \quad (\text{for some } \beta \in \mathbb{R})\end{aligned}$$

shows $\alpha = 0$. This is a contradiction and proves the lemma. $\square$

Lemma 5.2. *Let* $\bullet$ *be a complete left symmetric structure on* $\mathfrak{g}_3$*, with* $T(\mathfrak{g}_3) = 0$*, then* $\dim(\mathfrak{g}_{3,\infty}) \neq 4$.

Proof. Assume $\dim(\mathfrak{g}_{3,\infty}) = 4$. By Lemma 5.1 we have $\mathfrak{g}_{3,\infty} = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], R\rangle = \mathfrak{g}_3 \bullet \mathfrak{g}_3$ (with $R \in Z(\mathfrak{g}_3) \setminus [\mathfrak{g}_3, \mathfrak{g}_3]$). Hence we can find an element $X \in \mathfrak{g}_3 \setminus \mathfrak{g}_{3,\infty}$, with $\operatorname{codim}(C_{\mathfrak{g}_3}(X)) = 1$ such that $\mathfrak{g}_3 = \langle X\rangle \oplus \mathfrak{g}_{3,\infty}$. Define $X_3 = [e_1, X]$ and $X_4 = [e_1, X_3]$. (*Remark*: $[X, X_3] = 0$.)

The first step in the proof is to find an adapted basis for $\mathfrak{g}_3$. We already know that $\mathfrak{g}_{3,\infty} = \langle e_1, e_2, e_3, e_4\rangle = \langle e_1, R, X_3, X_4\rangle$, so we can take X as e_5. To find this basis we shall need following computations

$$X_3 \bullet \mathfrak{g}_3 = [e_1, X] \bullet \mathfrak{g}_3 \subseteq [e_1, X \bullet \mathfrak{g}_3] - X \bullet [e_1, \mathfrak{g}_3], \tag{4}$$

$$X_4 \bullet \mathfrak{g}_3 = [e_1, X_3] \bullet \mathfrak{g}_3 \subseteq [e_1, X_3 \bullet \mathfrak{g}_3] - X_3 \bullet [e_1, \mathfrak{g}_3]. \tag{5}$$

Note that a special case of (4), namely $X_3 \bullet X = [e_1, X \bullet X] - X \bullet [e_1, X]$, implies that $X_3 \bullet X \in [\mathfrak{g}_3, \mathfrak{g}_3]$.

Firstly we will show that $\dim(\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}) = 3$. As $X_4 = e_1 \bullet X_3$, $X_4 \in \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$. We compute $\mathfrak{g}_{3,\infty}$

$$\begin{aligned} \mathfrak{g}_{3,\infty} = (\mathfrak{g}_3 \bullet \mathfrak{g}_3) \bullet \mathfrak{g}_3 &= \bigl\langle e_1 \bullet \mathfrak{g}_3, [\mathfrak{g}_3, \mathfrak{g}_3] \bullet \mathfrak{g}_3, R \bullet \mathfrak{g}_3 \bigr\rangle \\ &= \langle X_3, \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}\rangle, \end{aligned}$$

where we used that $R \in Z(\mathfrak{g}_3)$ and $[\mathfrak{g}_3, \mathfrak{g}_3] \bullet \mathfrak{g}_3 \subseteq \mathfrak{g}_3 \bullet (\mathfrak{g}_3 \bullet \mathfrak{g}_3) = \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$. As $\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$ is a proper subset of $\mathfrak{g}_{3,\infty}$, we find $\dim(\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}) = 3$. Therefore $\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty} = \langle e_1, X_4, R + \alpha X_3\rangle$ (*Remark*: $X_3 \notin \mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty}$!).

Let us assume $\alpha \neq 0$. Then we have $X_4 = \frac{1}{\alpha} e_1 \bullet (R + \alpha X_3) \in \mathfrak{g}_3 \bullet (\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty})$. So, we find an adapted basis for $\mathfrak{g}_3$ with $e_2 = X_4$, $e_3 = R + \alpha X_3$, $e_4 = X_3$, $e_5 = X$. Then it follows from (4) and

$$\begin{aligned} R \bullet \mathfrak{g}_3 = \mathfrak{g}_3 \bullet R &= \mathfrak{g}_3 \bullet (R + \alpha X_3) - \alpha \mathfrak{g}_3 \bullet X_3 \\ &= \mathfrak{g}_3 \bullet e_3 - \alpha \mathfrak{g}_3 \bullet X_3 \end{aligned}$$

that $X_3 \bullet \mathfrak{g}_3 \subseteq \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3]\rangle = \langle e_1, e_2, e_4\rangle$ and also $R \bullet \mathfrak{g}_3 \subseteq \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3]\rangle = \langle e_1, e_2, e_4\rangle$. Therefore

$$\mathfrak{g}_{3,\infty} = \mathfrak{g}_{3,\infty} \bullet \mathfrak{g}_3 = \langle e_1, e_2, e_4\rangle.$$

This is a contradiction. Hence $\alpha = 0$ and $\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty} = \langle e_1, X_4, R\rangle$.

As last step to find an adapted basis for $\mathfrak{g}_3$ let us assume $X_4 \in \mathfrak{g}_3 \bullet (\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty})$. Therefore we can choose an adapted basis with $X_4 = e_2$, $R = e_3$, $X_3 = e_4$. This means that

$$\mathfrak{g}_{3,\infty} = \mathfrak{g}_{3,\infty} \bullet \mathfrak{g}_3 = \langle e_1, e_2, e_4\rangle,$$

which is again a contradiction, so $X_4 \notin \mathfrak{g}_3 \bullet (\mathfrak{g}_3 \bullet \mathfrak{g}_{3,\infty})$. So, finally we find an adapted basis with $e_2 = R$, $e_3 = X_4$, $e_4 = X_3$ and $e_5 = X$.

In the next part of the proof we will find an element of $\mathfrak{g}_3$ which belongs to $T(\mathfrak{g}_3)$. This contradicts $T(\mathfrak{g}_3) = 0$.

When we compute $\mathfrak{g}_{3,\infty} \bullet \mathfrak{g}_3$ by using (4) and (5) we find

$$\mathfrak{g}_{3,\infty} = \mathfrak{g}_{3,\infty} \bullet \mathfrak{g}_3 = \langle e_1, e_3, e_4, X \bullet X_4\rangle.$$

As $X_4 = e_3$, we have $X \bullet X_4 = a e_1 + b R$, $a \in \mathbb{R}, b \in \mathbb{R}_0$ ($\dim(\mathfrak{g}_{3,\infty}) = 4$!). Now we can compute the following products by using (4) and (5) and the fact that $\mathfrak{g}_3 \bullet R \in \langle e_1\rangle$:

$$X_3 \bullet R = 0, \qquad X_4 \bullet R = 0, \qquad X_3 \bullet X_4 = 0, \qquad X_4 \bullet X_4 = 0.$$

Hence $X_4 \bullet (\mathfrak{g}_3 \bullet \mathfrak{g}_3) = X_4 \bullet \langle e_1, R, X_4, X_3\rangle = 0$. Finally we compute

$$\begin{aligned}
X \bullet (X_4 \bullet X) &= bX \bullet R \\
&= X_4 \bullet (X \bullet X) = 0, \\
R \bullet (X_4 \bullet X) &= bR \bullet R \\
&= X_4 \bullet (X \bullet R) = 0.
\end{aligned}$$

As $b \neq 0$, we find $R \bullet X = 0$ and $R \bullet R = 0$. We knew already that $R \bullet [\mathfrak{g}_3, \mathfrak{g}_3] = 0$. This means $R \in T(\mathfrak{g}_3)$, which is a contradiction. $\square$

Now, we can look for those products $\bullet$ such that $T(\mathfrak{g}_3) = 0$. From Lemma 5.2 we already know that $\dim(\mathfrak{g}_{3,\infty}) = 3$, and as $e_1 \in \mathfrak{g}_{3,\infty}$, $[\mathfrak{g}_3, \mathfrak{g}_3] = [e_1, \mathfrak{g}_3] \subseteq \mathfrak{g}_{3,\infty}$. Moreover $\mathfrak{g}_{3,\infty} = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3] \rangle$. We also know $\mathfrak{g}_{3,\infty} \subseteq \mathfrak{g}_3 \bullet \mathfrak{g}_3 \subseteq \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], R \rangle$ (with R as in Lemma 5.1). Hence we have two possibilities: either $\mathfrak{g}_{3,\infty} = \mathfrak{g}_3 \bullet \mathfrak{g}_3$ or $\mathfrak{g}_{3,\infty} \subsetneq \mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle e_1, [\mathfrak{g}_3, \mathfrak{g}_3], R \rangle$.

Writing out the possible products, we come to following conclusion:

For the Lie algebra $\mathfrak{g}_3$, we can find a vector space basis $\langle X_1, X_2, X_3, X_4, X_5 \rangle$ with $[X_1, X_2] = X_3$ and $[X_1, X_3] = X_4$ (the other brackets are zero), such that the products in a complete left symmetric structure with trivial centre are given by:

Case 1: $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle X_1, X_3, X_4 \rangle$

$$\begin{array}{lll}
X_1 \bullet X_1 = 0, & X_3 \bullet X_1 = 0, & X_4 \bullet X_1 = 0, \\
X_1 \bullet X_2 = X_3, & X_3 \bullet X_2 = 0, & X_4 \bullet X_2 = aX_1, \\
X_1 \bullet X_3 = X_4, & X_3 \bullet X_3 = -aX_1, & X_4 \bullet X_3 = 0, \\
X_1 \bullet X_4 = 0, & X_3 \bullet X_4 = 0, & X_4 \bullet X_4 = 0, \\
X_1 \bullet X_5 = 0, & X_3 \bullet X_5 = 0, & X_4 \bullet X_5 = 0, \\
X_5 \bullet X_1 = 0, & X_2 \bullet X_1 = 0, & \\
X_5 \bullet X_2 = bX_1, & X_2 \bullet X_2 = cX_1 + dX_4, & \\
X_5 \bullet X_3 = 0, & X_2 \bullet X_3 = 0, & \\
X_5 \bullet X_4 = 0, & X_2 \bullet X_4 = aX_1, & \\
X_5 \bullet X_5 = eX_1, & X_2 \bullet X_5 = bX_1, &
\end{array} \tag{6}$$

with $a, e \in \mathbb{R}_0$ and $b, c, d \in \mathbb{R}$.

Case 2: $\mathfrak{g}_3 \bullet \mathfrak{g}_3 = \langle X_1, X_3, X_4, X_5 \rangle$

$$\begin{array}{lll}
X_1 \bullet X_1 = 0, & X_3 \bullet X_1 = 0, & X_4 \bullet X_1 = 0, \\
X_1 \bullet X_2 = X_3, & X_3 \bullet X_2 = 0, & X_4 \bullet X_2 = aX_1, \\
X_1 \bullet X_3 = X_4, & X_3 \bullet X_3 = -aX_1, & X_4 \bullet X_3 = 0, \\
X_1 \bullet X_4 = 0, & X_3 \bullet X_4 = 0, & X_4 \bullet X_4 = 0, \\
X_1 \bullet X_5 = 0, & X_3 \bullet X_5 = 0, & X_4 \bullet X_5 = 0, \\
X_5 \bullet X_1 = 0, & X_2 \bullet X_1 = 0, & \\
X_5 \bullet X_2 = bX_1 + cX_4, & X_2 \bullet X_2 = eX_1 + fX_4 + dX_5, & \\
X_5 \bullet X_3 = 0, & X_2 \bullet X_3 = 0, & \\
X_5 \bullet X_4 = 0, & X_2 \bullet X_4 = aX_1, & \\
X_5 \bullet X_5 = \frac{ac}{d} X_1, & X_2 \bullet X_5 = bX_1 + cX_4, &
\end{array} \tag{7}$$

with $a, c, d \in \mathbb{R}_0$ and $b, e, f \in \mathbb{R}$.

Just as in the case of $\mathfrak{g}_5$, some easy (but lengthy) computations allow one to determine all the isomorphism types of the left symmetric structures given above. We restrict ourselves to formulating the result.

Theorem 5.3. *There are* 6 *isomorphism types of complete left symmetric structures* $\bullet$ *with trivial centre on the Lie algebra* $\mathfrak{g}_3$. *For* 4 *of them,* $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ *is* 3*-dimensional and these classes are given by the product determined by the formulas* (6) *and for the following parameters*:

a	b	c	d	e
1	0	0	0	1
1	0	0	0	−1
1	0	0	1	1
1	0	0	1	−1

For the other 2 *structures,* $\mathfrak{g}_3 \bullet \mathfrak{g}_3$ *is* 4*-dimensional and the products are determined by* (7) *with parameters*:

a	b	c	d	e	f
1	0	1	1	0	0
1	0	1	−1	0	0

5.2. Centerless left symmetric structures on $\mathfrak{g}_6$

Finally, we have to take a look at the Lie algebra $\mathfrak{g}_6$. For this Lie algebra we will only give the classification of all complete left symmetric structures with trivial centre. Analogously to the previous cases, one finds that there are three types of complete left symmetric structures $\bullet$ with trivial centre.

These are given as follows:
For the Lie algebra $\mathfrak{g}_6$, there is a vector space basis $\langle X_1, X_2, X_3, X_4, X_5\rangle$ with $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$ and $[X_2, X_5] = X_4$ (the other brackets are zero), such that the complete left symmetric structures with trivial centre on this Lie algebra are given by following products:

Case 1: $\mathfrak{g}_{6,\infty} = \mathfrak{g}_6 \bullet \mathfrak{g}_6 = \langle X_1, X_3, X_4\rangle$

$$\begin{array}{lll}
X_1 \bullet X_1 = 0, & X_2 \bullet X_1 = 0, & X_3 \bullet X_1 = 0,\\
X_1 \bullet X_2 = X_3, & X_2 \bullet X_2 = dX_1 + eX_4, & X_3 \bullet X_2 = 0,\\
X_1 \bullet X_3 = X_4, & X_2 \bullet X_3 = 0, & X_3 \bullet X_3 = -aX_1,\\
X_1 \bullet X_4 = 0, & X_2 \bullet X_4 = aX_1, & X_3 \bullet X_4 = 0,\\
X_1 \bullet X_5 = 0, & X_2 \bullet X_5 = cX_1 + 2X_4, & X_3 \bullet X_5 = 0,\\
X_4 \bullet X_1 = 0, & X_5 \bullet X_1 = 0, & \\
X_4 \bullet X_2 = aX_1, & X_5 \bullet X_2 = cX_1 + X_4, & \\
X_4 \bullet X_3 = 0, & X_5 \bullet X_3 = 0, & \\
X_4 \bullet X_4 = 0, & X_5 \bullet X_4 = 0, & \\
X_4 \bullet X_5 = 0, & X_5 \bullet X_5 = bX_1, &
\end{array} \tag{8}$$

with $a \in \mathbb{R}_0$ and $b, c, d, e \in \mathbb{R}$.

Case 2: $\mathfrak{g}_{6,\infty} = \langle X_2, X_3, X_4\rangle \subset \mathfrak{g}_6 \bullet \mathfrak{g}_6 = \langle X_2, X_3, X_4, X_5\rangle$

$$\begin{array}{lll}
X_2 \bullet X_1 = -X_3, & X_3 \bullet X_1 = 0, & X_4 \bullet X_1 = aX_2,\\
X_2 \bullet X_2 = 0, & X_3 \bullet X_2 = 0, & X_4 \bullet X_2 = 0,\\
X_2 \bullet X_3 = 0, & X_3 \bullet X_3 = 0, & X_4 \bullet X_3 = 0,\\
X_2 \bullet X_4 = 0, & X_3 \bullet X_4 = 0, & X_4 \bullet X_4 = 0,\\
X_2 \bullet X_5 = X_4, & X_3 \bullet X_5 = aX_2, & X_4 \bullet X_5 = 0,
\end{array}$$
$$\begin{array}{ll}
X_5 \bullet X_1 = fX_2 + gX_4, & X_1 \bullet X_1 = bX_2 + cX_4 + dX_3 - X_5,\\
X_5 \bullet X_2 = 0, & X_1 \bullet X_2 = 0,\\
X_5 \bullet X_3 = aX_2, & X_1 \bullet X_3 = X_4,\\
X_5 \bullet X_4 = 0, & X_1 \bullet X_4 = aX_2,\\
X_5 \bullet X_5 = a(d-g)X_2, & X_1 \bullet X_5 = fX_2 + gX_4,
\end{array} \tag{9}$$

with $a \in \mathbb{R}_0$ and $b, c, d, f, g \in \mathbb{R}$.

Case 3: $\mathfrak{g}_{6,\infty} = \langle X_1, X_3, X_4\rangle \subset \mathfrak{g}_6 \bullet \mathfrak{g}_6 = \langle X_1, X_3, X_4, X_5\rangle$

$$\begin{array}{lll}
X_1 \bullet X_1 = 0, & X_3 \bullet X_1 = 0, & X_4 \bullet X_1 = 0,\\
X_1 \bullet X_2 = X_3, & X_3 \bullet X_2 = 0, & X_4 \bullet X_2 = aX_1,\\
X_1 \bullet X_3 = X_4, & X_3 \bullet X_3 = -aX_1, & X_4 \bullet X_3 = 0,\\
X_1 \bullet X_4 = 0, & X_3 \bullet X_4 = 0, & X_4 \bullet X_4 = 0,\\
X_1 \bullet X_5 = 0, & X_3 \bullet X_5 = 0, & X_4 \bullet X_5 = 0,
\end{array}$$
$$\begin{array}{ll}
X_5 \bullet X_1 = 0, & X_2 \bullet X_1 = 0,\\
X_5 \bullet X_2 = cX_1 + \left(1 + \frac{fb}{a}\right)X_4, & X_2 \bullet X_2 = dX_1 + eX_4 + fX_5,\\
X_5 \bullet X_3 = 0, & X_2 \bullet X_3 = 0,\\
X_5 \bullet X_4 = 0, & X_2 \bullet X_4 = aX_1,\\
X_5 \bullet X_5 = bX_1, & X_2 \bullet X_5 = cX_1 + \left(2 + \frac{fb}{a}\right)X_4,
\end{array} \tag{10}$$

with $a, f \in \mathbb{R}_0$ and $b, c, d, e \in \mathbb{R}$.

Theorem 5.4. *Up to isomorphism, there are infinitely many complete left symmetric structures* $\bullet$ *on* $\mathfrak{g}_6$ *with trivial centre.*

For 3 *of these structures,* $\mathfrak{g}_{6,\infty} = \mathfrak{g}_6 \bullet \mathfrak{g}_6$ *are* 3*-dimensional and the product is given by* (8) *with parameters*:

a	b	c	d	e
1	0	0	0	0
1	1	0	0	0
1	−1	0	0	0

For 3 *other classes,* $\mathfrak{g}_{6,\infty}$ *is* 3*-dimensional and* $\mathfrak{g}_6 \bullet \mathfrak{g}_6$ *is* 4*-dimensional. The product is given by* (9) *for the parameters*:

a	b	c	d	f	g
1	0	0	0	0	0
1	0	0	0	1	0
1	0	0	0	0	1

Finally, there are also 4 *one-parameter families of such left symmetric structures with* $\mathfrak{g}_{6,\infty}$ 3*-dimensional and* $\mathfrak{g}_6 \bullet \mathfrak{g}_6$ 4*-dimensional. The products in these cases are given by* (10) *for the parameters*:

a	b	c	d	e	f
1	−1	0	0	e	1
1	1	0	0	e	−1
1	b	0	0	0	1
1	b	0	0	0	−1

with $b, e \in \mathbb{R}$ *and* $e \geqslant 0$.

6. Conclusion

In this paper we gave a classification of all simply transitive affine actions of five-dimensional nilpotent Lie groups, without non-trivial pure translations. The only Lie groups admitting such actions are G_3, G_5, G_6 (with G_i the Lie group corresponding with the Lie algebra $\mathfrak{g}_i$). For G_5 there are five isomorphism classes, for G_3 six isomorphism classes and finally, for G_6 we found six isomorphism classes and four infinite families depending on one parameter. So for G_6 we find the (rather surprising) result that there are infinitely many non-isomorphic simply transitive affine actions without translations!

For the Lie groups G_1, G_2, G_4, G_8, G_9 we already knew that there are no such actions possible. In this paper we have proven that also G_7 does not admit any such an action.

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